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EXTREMUM PROPERTIES OF FINITE-STEP SOLUTIONS IN ELASTOPLASTICITY WITH NONLINEAR MIXED HARDENING

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Abstract—The class of elastic–plastic constitutive laws assumed herein can be described as follows. We envisage $y \geq 1$ yield criteria (or "modes") which define the current yield surface; each yield function ϕ_α consists of two addends: an effective stress Φ_α and a yield limit Y_α ($\alpha = 1, \ldots, y$). The former is an order-one, positively homogeneous function of the difference between the stresses σ_{ij} and reference stresses α_{ij}, which are generally nonlinear functionals of the plastic strain history, so that kinematic hardening is accounted for. The yield limit Y_α is generally a nonlinear function of y nondecreasing internal variables λ_β, so that another hardening mechanism (which may reduce to isotropic hardening) is accommodated in the model. The rates of these variables play the role of plastic multipliers in the flow rule.

A finite step in the geometrically linear evolutive analysis of such solids is defined according to the backward-difference (or "stepwise holonomic") strategy for approximate time integration. For the "finite-step" boundary value problem thus arising, various extremum characterizations of solutions are established and the underlying constitutive restrictions are pointed out and discussed.

1. INTRODUCTION

Extremum theorems for the incremental (rate) solution to boundary value problems in elastoplasticity, and for the path-independent solution in the "deformation theory" of plasticity, have been the subjects of a fairly abundant literature. The earlier part of this is synthesized in treatises such as Martin's (1975), the more recent part is still disseminated in research papers, while broad mathematical foundations and alternative characterizations of solutions (in terms of variational inequalities) can be found in comprehensive works by Panagiotopoulos (1985) and Sewell (1987).

Some of the early contributions to extremum properties in plasticity, precisely those concerning constitutive models with linear yield functions and linear hardening (Maier, 1969, 1970a), have had fruitful implications on the computational aspects of inelastic analysis. In fact, for those models, rate problems, path-independent ("holonomic", single-step) analysis and finite-step (stepwise-holonomic) problems within marching solution methods, all become amenable to and solvable by quadratic programming (after suitable space discretization). The stepwise-holonomic method (now more frequently called "backward difference" method) for approximate time integration in elastic–plastic analysis was proposed and developed by De Donato and Maier (1972, 1973) in such a context of plasticity with piecewise-linearized yield criteria. Mechanically interpreted, this procedure basically rests on two provisions: within each load step the elastic–plastic, path-dependent (nonholonomic), stress–strain law is replaced by a path-independent (holonomic) law generated from it. Between two successive steps, all quantities which depend on the yielding history are updated, thus accounting for the intrinsic irreversibility of plasticity. In the context of piecewise-linearized plasticity, stepwise holonomic elastic–plastic analysis exhibits the appealing feature of being exposed only to errors entailed by the constitutive idealization, and by possible local unloading within a single load step (no error accumulation, unconditional stability). Surveys of the literature on this area of plasticity can be found in Cohn *et al.* (1979) and Maier *et al.* (1982). These surveys, however, do not cover more recent contributions such as those proposed by Dittmer *et al.* (1985), Martin *et al.* (1987), Franchi and Genna (1987) and Zouain *et al.* (1988).

In the last few years, methods for the approximate time integration of the (nonlinear, differential) relationships of elastoplasticity have become a fashionable research topic in computational mechanics. Valuable contributions are due to Ortiz and Popov (1985), Simo and Taylor (1986), Matthies and Strang (1979), Simo *et al.* (1988), Casciaro and Mancuso (1988) and to other authors. The stepwise-holonomic or backward difference time integration procedure is actively investigated, *in primis* by J. B. Martin and his coworkers.

In this paper, the finite-step boundary value problem arising in stepwise-holonomic elastoplastic analysis is addressed with reference to a category of material models (described in Section 2), endowed with mixed nonlinear hardening. Namely, as a consequence of yielding processes the elastic domain translates in the stress space (kinematic hardening) and the yield limits change (isotropic hardening, called so here as it may describe changes in size but not in shape of the yield domain). For the above boundary value problem in finite increments (but in the small deformations, geometrically linear range), we derive in Sections 3–4 extremum properties of solutions. Constitutive hypotheses found to be sufficient for the validity of some of them are discussed in Section 2. In a parallel paper (Comi and Maier, 1990), one of the present results is shown to provide a convergence criterion for a popular iterative solution scheme, thus substantiating their potential use in computational plasticity.

In general terms, whenever suitable, space discretizations (e.g. finite element models) are adopted, the extremum theorems derived in what follows make the step problem amenable to a problem in nonlinear programming.

The basic purpose of the present paper is to extend to nonlinear multiple yield criteria and nonlinear mixed hardening, the aforementioned earlier work on piecewise-linearized plastic models.

Such an extension appears to be supplementary to the contributions so far available on the subject, to the authors' knowledge, such as those proposed by Franchi and Genna (1984), Feijòo and Zouain (1988), Reddy and Griffin (1986) and Maier and Novati (1990).

As for the notation and analytical dress, the developments in what follows are carried out by the conventional formalism of classical plasticity (rather than by the more elegant but also more abstract mathematical formalism of nonsmooth mechanics or convex analysis).

The usual Cartesian tensor description is used, with the index summation convention (when this does not apply, repeated indices are avoided and a single subscript is used for the parentheses enclosing the factors). A factor in parentheses is preceded by a dot denoting product when it might be confused with the argument of a functional dependence.

2. PROBLEM FORMULATIONS: INCREMENTAL VS STEPWISE HOLONOMIC CONSTITUTIVE LAWS

2.1. *A class of material models*

The most general kind of elastoplastic constitutions considered herein, are specified by the following relation set, in an orthogonal reference x_i ($i = 1, 2, 3$):

$$\sigma_{ij} = E_{ijrs}\varepsilon^{e}_{rs} = E_{ijrs}(\varepsilon_{rs} - \varepsilon^{p}_{rs} - \theta_{rs})\,; \qquad (1a, b)$$

$$\phi_\alpha = \Phi_\alpha(\sigma_{rs} - \alpha_{rs}(\varepsilon^{p}_{lm})) - Y_\alpha(\lambda_\beta) \leqslant 0 \quad (\alpha, \beta = 1, \ldots, y)\,; \qquad (2a, b)$$

$$\dot{\varepsilon}^{p}_{ij} = \frac{\partial \psi_\alpha}{\partial \sigma_{ij}}(\sigma_{rs} - \alpha_{rs}(\varepsilon^{p}_{lm}))\dot{\lambda}_\alpha, \quad \dot{\lambda}_\alpha \geqslant 0\,; \qquad (3a, b)$$

$$\phi_\alpha \dot{\lambda}_\alpha = 0 \quad \text{or} \quad (\phi\dot{\lambda})_\alpha = 0. \qquad (4a, b)$$

Here, and henceforth, dots mark time derivatives; the customary properties (positive definiteness and symmetries) of the moduli tensor of linear elasticity E_{ijrs} are assumed. Equation (1b) expresses the additivity of elastic (ε^{e}_{rs}), plastic (ε^{p}_{rs}) and imposed (e.g. thermal, θ_{rs}) strains. Equations (2) define the current elastic domain (and the current yield surface as its boundary) in the stress space by y yield functions ϕ_α, each of which is expressed as the sum of an "effective stress" Φ_α, and a yield limit Y_α. The former is seen to depend on plastic strains so that a translation of the yield surface occurs at yielding and is defined by the "reference stresses" α_{ij} (kinematic hardening). The dependence of the latter addend Y_α on the variables λ_β will be called "isotropic hardening", although the shape of the elastic domain is preserved only in special cases, e.g. for a single mode ($y = 1$) and if Φ_α is a first-order homogeneous function. The role of the nondecreasing variables λ_α is specified by eqns (2)–(4): their time derivatives act as "plastic multipliers" in eqn (3a) and are related to the yield functions by the complementarity equation (4). By virtue of the sign constraints on the variables involved, eqn (4) holds also component-wise for each α, and implies that:

$$\phi_\alpha \dot{\lambda}_\alpha = 0 \quad \text{or} \quad (\phi\dot{\lambda})_\alpha = 0, \tag{5a, b}$$

which expresses Prager's consistency rule for "loading" (yielding) and "unloading" processes. The variables λ_α are thought of as measures, at the phenomenological macroscale level, of irreversible rearrangements occurring at the microscale level inside the material and, hence, will be called "internal variables" in what follows, e.g. Martin (1981). The above set of relations is homogeneous in time t: this implies the time-independent ("inviscid") character of the material behaviour described and thus reduce t to an event-ordering variable.

2.2. *Stepwise holonomic constitutive laws*

Assume a situation in which all variables are known (and will be marked by barred symbols). Starting from this situation, consider finite increments (denoted by Δ; finite in contrast to infinitesimal or rates) and relate them by the following relation set:

$$\Delta\sigma_{ij} = E_{ijrs} \cdot (\Delta\varepsilon_{rs} - \Delta\varepsilon^{p}_{rs} - \Delta\theta_{rs}); \tag{6}$$

$$\phi_\alpha = \Phi_\alpha(\bar{\sigma}_{rs} + \Delta\sigma_{rs} - \alpha_{rs}(\bar{\varepsilon}^{p}_{lm} + \Delta\varepsilon^{p}_{lm})) - Y_\alpha(\bar{\lambda}_\beta + \Delta\lambda_\beta) \leqslant 0; \tag{7a, b}$$

$$\Delta\varepsilon^{p}_{ij} = \frac{\partial\psi_\alpha}{\partial\sigma_{ij}}(\bar{\sigma}_{rs} + \Delta\sigma_{rs} - \alpha_{rs}(\bar{\varepsilon}^{p}_{lm} + \Delta\varepsilon^{p}_{lm}))\Delta\lambda_\alpha, \quad \Delta\lambda_\alpha \geqslant 0; \tag{8a, b}$$

$$\phi_\alpha \Delta\lambda_\alpha = 0 \quad (\alpha, \beta = 1, \ldots, y). \tag{9}$$

Equations (6)–(9) will be called the stepwise–holonomic laws (path-independent within the step), generated by the elastoplastic incremental (nonholonomic, path-dependent) constitution (1)–(4), to which they are easily seen to reduce for $\Delta t \to 0 (\Delta \to d)$. Clearly, these relations are algebraic, nonlinear, while eqns (1)–(4) are differential nonlinear.

2.3. *Finite-step boundary value problems*

Consider a solid which occupies a region V (in R^3), with a smooth boundary S, where n_i denotes the outward normal direction. The evolutive response is sought to a given history of external actions: body forces $b_i(t)$ and imposed strains $\theta_{ij}(t)$ in V; tractions $T_i(t)$ on S_T; imposed displacements $\bar{u}_i(t)$ on S_u (S_T and S_u being complementary parts of S). The compatibility and equilibrium equations are assumed to be unaffected by geometry changes. These equations associated with the constitutive eqns (1)–(4) give rise to the set of governing relations, i.e. to the b.v. problem in rates. An approximate marching solution of this nonlinear differential equation set is usually based on a preliminary sub-division of the external action history into the sequence of loading conditions at instants $t_0 = 0$, t_1, $t_2, \ldots, t_n = t_{n-1} + \Delta t_n$. These are chosen in such a way that the external action variation over each time interval Δt_i is not far from linear.

Suppose that all the variable fields are known throughout V at instant $\bar{t} \equiv t_{n-1}$ (their values will be represented by barred symbols). Denoting by Δ increments over $\Delta t \equiv \Delta t_n$ of both input data and response variables, let us associate the stepwise–holonomic constitutive laws (6)–(9) to the geometric compatibility and equilibrium equations:

$$\Delta\varepsilon_{ij} = \tfrac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i}) \quad \text{in } V, \quad \Delta u_i = \Delta\bar{u}_i \quad \text{on } S_u, \tag{10a, b}$$

$$\Delta\sigma_{ij,i}+\Delta\bar{b}_j = 0 \quad \text{in } V, \quad \Delta\sigma_{ij}n_i = \Delta\bar{T}_j \quad \text{on } S_T. \tag{11a, b}$$

The finite-step b.v. problem governed by eqns (6)–(11) arises when the approximate time integration of the original b.v. problem in rates, is formulated according to an Euler, backward difference, fully implicit scheme. The procedure for each step defines all state-dependent variables and all derivatives in the unknown state at the step end, and enforces the algebraic, plastic relations there [eqns (7)–(9)]. Allowance for the irreversible, history-dependent nature of plasticity is made from step to step, by updating the internal variables λ_α with sign-constrained increments.

As mentioned in Section 1, stepwise–holonomic (or backward difference) time-integration in the above sense, leading to the b.v. problem (6)–(11), is the subject of abundant recent and nonrecent literature. Here, this finite-step problem will be studied (in Sections 3, 4, 5) as for the extremum properties of solutions, under some of the restrictions to be considered next.

2.4. *Constitutive restrictions*

The following hypotheses which specialize the plastic constitutive law (2)–(4), have a significant role in the subsequent discussions.

(a) The effective stresses are differentiable convex functions of their argument (denoted by τ_{ij}), namely:

$$\Phi_\alpha(\tau_{ij}) \geqslant \Phi_\alpha(\tau'_{ij})+\frac{\partial\Phi_\alpha}{\partial\tau_{rs}}(\tau'_{ij})\cdot(\tau_{rs}-\tau'_{rs}), \quad \text{for any } \tau_{ij}, \tau'_{ij}. \tag{12}$$

This hypothesis is equivalent to the convexity of the yield functions ϕ_α, with respect to stresses and hence, entails the convexity of all current elastic domains, which is one of the requirements of Drucker's stability postulate (Drucker, 1960; 1988).

(b) The effective stresses are positively homogeneous functions of order one of their argument τ_{ij}, i.e.

$$\Phi_\alpha(c\tau_{ij}) = c\Phi_\alpha(\tau_{ij}), \quad \text{for any } c \geqslant 0. \tag{13}$$

By Euler's theorem and by an easily proved consequence of it, property (13) implies that

$$\Phi_\alpha(\tau_{ij}) = \frac{\partial\Phi_\alpha}{\partial\tau_{rs}}(\tau_{ij})\tau_{rs}; \quad \frac{\partial^2\Phi_\alpha}{\partial\tau_{rs}\,\partial\tau_{hk}}(\tau_{ij})\tau_{hk} = 0. \tag{14a, b}$$

Note that eqn (14b) is satisfied whenever the argument τ_{ij} coincides with the factor τ_{hk}. Clearly, the latter tensor τ_{hk} can be interpreted as a solution of the homogeneous linear equation system whose coefficient matrix is the Hessian of Φ_α, evaluated for any fixed τ_{ij}. Most plastic constitutive laws in use comply with this hypothesis, or can be re-formulated so that they comply with it.

(c) The yield functions ϕ_α and the plastic potentials ψ_α are identical, namely:

$$\psi_\alpha = \phi_\alpha. \tag{15}$$

This association of the flow rule with the yield criterion or normality property, required by Drucker's stability postulate, is known to be violated by important categories of materials such as geomaterials and concrete.

(d) The plastic work (per unit volume) is a functional of the plastic strain history:

$$W^p(t) = \int_0^t \sigma_{ij}(t')\dot{\varepsilon}^p_{ij}(t')\,dt'. \tag{16}$$

When the constitutive restrictions (b) and (c) are satisfied, i.e. eqns (13) and (15) hold, then through eqns (3a) and (4), expression (16) for W^p can be given the form:

$$W^p(t) = \Pi(t) + \Gamma(t), \tag{17}$$

$$\Pi(t) \equiv \int_0^t Y_\alpha(t')\dot{\lambda}_\alpha(t')\,dt'; \quad \Gamma(t) \equiv \int_0^t \alpha_{ij}(t')\dot{\varepsilon}^p_{ij}(t')\,dt'. \tag{18a, b}$$

The hypothesis, called henceforth "reciprocal hardening", means that:

$$\frac{\partial Y_\alpha}{\partial \lambda_\beta} = \frac{\partial Y_\beta}{\partial \lambda_\alpha}; \quad \frac{\partial \alpha_{ij}}{\partial \varepsilon^p_{rs}} = \frac{\partial \alpha_{rs}}{\partial \varepsilon^p_{ij}}. \tag{19a, b}$$

This symmetry of Jacobian matrices is a necessary and sufficient condition for the path-independence of the two plastic work addends (18): the former, associated to the so-called isotropic hardening, becomes a function of the internal variables only $\Pi(\lambda_\alpha)$; the latter, associated to the kinematic hardening, becomes a function of the plastic strains only $\Gamma(\varepsilon^p_{ij})$. Such reciprocity in the interaction between yield modes is violated by some material models (e.g. by the three-mode model due to Resende and Martin, 1985).

(e) When the preceding hypothesis is complied with, a further restriction is the convexity of functions $\Pi(\lambda_\alpha)$ and $\Gamma(\varepsilon^p_{ij})$, i.e. the positive, semi-definiteness of their Hessian matrices which, clearly, coincide with the (symmetric) Jacobian matrices (19). Using eqns (18) and (19), the second-order plastic work can be expressed in the form:

$$\delta^{(2)} W^p = \frac{1}{2}\delta Y_\alpha \delta\lambda_\alpha + \frac{1}{2}\delta\alpha_{ij}\delta\varepsilon^p_{ij} = \frac{1}{2}\delta\lambda_\alpha \frac{\partial Y_\alpha}{\partial \lambda_\beta}\delta\lambda_\beta + \frac{1}{2}\delta\varepsilon^p_{ij}\frac{\partial \alpha_{ij}}{\partial \varepsilon^p_{rs}}\delta\varepsilon^p_{rs}. \tag{20}$$

Equation (20) shows that by the present hypothesis, only non-negative work can be done by an external agency, causing an infinitesimal perturbation while preserving equilibrium. This means material stability by Hill's criterion (Hill, 1957); it is one of the requirements of Drucker's postulate and rules out softening behaviour.

3. MINIMUM PROPERTIES OF SOLUTIONS IN THE FINITE INCREMENTS OF KINEMATIC VARIABLES

In this section, we address the b.v. problem of Section 2.3, defined by eqns (6)–(11) and prove two extremum characterizations of its solution on the basis of all the constitutive restrictions (a)–(e), discussed in Section 2.4.

3.1. *Kinematic (potential energy) theorem*

The following statement can be regarded as a manifestation of the potential energy principle in the present context of incremental elastoplasticity, discretized in time in the sense of Sections 2.2 and 2.3.

Proposition 1. Assuming the constitutive restrictions (a)–(e) of Section 2.4, consider the optimization problem:

$$\min \Big\{ \Omega(\Delta u_i, \Delta\lambda_\alpha, \Delta\varepsilon^e_{ij}, \Delta\varepsilon^p_{ij}) \equiv \frac{1}{2}\int_V \Delta\varepsilon^e_{ij} E_{ijhk} \Delta\varepsilon^e_{hk}\,dV + \int_V \Pi(\bar{\lambda}_\alpha + \Delta\lambda_\alpha)\,dV$$
$$+ \int_V \Gamma(\bar{\varepsilon}^p_{ij} + \Delta\varepsilon^p_{ij})\,dV + \int_V \bar{\sigma}_{ij}\Delta\varepsilon^e_{ij}\,dV - \int_V (\bar{b}_i + \Delta\bar{b}_i)\Delta u_i\,dV - \int_{S_T} (\bar{T}_i + \Delta\bar{T}_i)\Delta u_i\,dS \Big\}, \tag{21}$$

subject to the relations ("constraints"):

$$\Delta\varepsilon^e_{ij} = \tfrac{1}{2}(\Delta u_{j,i} + \Delta u_{i,j}) - \Delta\varepsilon^p_{ij} - \Delta\bar{\vartheta}_{ij} \quad \text{in } V; \quad \Delta u_i = \Delta\bar{u}_i \quad \text{on } S_u; \tag{22a, b}$$

$$\Delta\varepsilon^{p}_{ij} = \frac{\partial\Phi_\alpha}{\partial\sigma_{ij}}(E_{hk\ell m}\cdot(\bar{\varepsilon}^{e}_{\ell m}+\Delta\varepsilon^{e}_{\ell m})-\alpha_{hk}(\bar{\varepsilon}^{p}_{\ell m}+\Delta\varepsilon^{p}_{\ell m}))\Delta\lambda_\alpha, \quad \Delta\lambda_\alpha \geqslant 0 \quad \text{in } V. \tag{23a, b}$$

The fact that a field Δu_i, $\Delta\lambda_\alpha$, $\Delta\varepsilon^{e}_{ij}$ minimizes the functional Ω (21), subject to the constraints (22)–(23), is a sufficient and necessary condition for solving the finite-step problem defined by eqns (6)–(11); for the given external action increments $\Delta\bar{b}_i$, $\Delta\bar{u}_i$, $\Delta\bar{\theta}_{ij}$, starting from a known situation (barred symbols).

Proof of sufficiency. Consider the Lagrangian functional:

$$L(\Delta u_i, \Delta\lambda_\alpha, \Delta\varepsilon^{e}_{ij}, \Delta\varepsilon^{p}_{ij}, \ell_\alpha; \xi_{ij}, \beta_{ij}, \eta_\alpha) \equiv \Omega + \int_V \xi_{ij}\cdot[\Delta\varepsilon^{e}_{ij} - \text{r.h.s. (22a)}]\,dV$$

$$+\int_V \beta_{ij}\cdot[\Delta\varepsilon^{p}_{ij} - \text{r.h.s. (23a)}]\,dV + \int_V \eta_\alpha\cdot(\Delta\lambda_\alpha - \ell^2_\alpha)\,dV; \tag{24}$$

where ξ_{ij}, β_{ij} and η_α are Lagrange multipliers, r.h.s. $(\cdot)$ means right-hand side of eqn $(\cdot)$ for brevity, and eqn (23b) has been interpreted as $\Delta\lambda_\alpha = \ell^2_\alpha$.

Mark by a tilde a solution of the optimization problem considered (and, for brevity, also the cumulative values; e.g. $\tilde{\lambda}_\alpha = \bar{\lambda}_\alpha + \Delta\tilde{\lambda}_\alpha$), calculate the first variation of L around this solution and, setting:

$$\tau_{hk} \equiv E_{hkrs}\varepsilon^{e}_{rs} - \alpha_{hk}(\varepsilon^{p}_{rs}) = \sigma_{hk} - \alpha_{hk}(\varepsilon^{p}_{rs}), \tag{25}$$

express its vanishing for all variations of the independent variables listed in eqn (24):

$$\delta^{(1)}L = \delta^{(1)}\Omega + \int_V \xi_{ij}\{\delta\varepsilon^{e}_{ij} - \delta[\text{r.h.s. (22a)}]\}\,dV$$

$$+\int_V \beta_{ij}\cdot\left[\delta\varepsilon^{p}_{ij} - \frac{\partial\Phi_\alpha}{\partial\sigma_{ij}}(\tilde{\tau}_{hk})\delta\lambda_\alpha - \Delta\tilde{\lambda}_\alpha\frac{\partial^2\phi_\alpha}{\partial\sigma_{ij}\,\partial\sigma_{rs}}(\tilde{\tau}_{hk})\delta\tau_{rs}\right]dV + \int_V \eta_\alpha\cdot[\delta\lambda_\alpha - 2(\tilde{\ell}\delta\ell)_\alpha])\,dV = 0. \tag{26}$$

The stationarity of L with respect to the Lagrangian multipliers is another way of enforcing the constraints (22)–(23), being understood that the boundary compatibility (22b) is tacitly required as $\delta u_i = 0$ on S_u. The stationarity (26) with respect to the variations δu_i, $\delta\varepsilon^{e}_{ij}$, $\delta\varepsilon^{p}_{ij}$, $\delta\lambda_\alpha$ and $\delta\ell_\alpha$ in turn, gives rise to the following equations, respectively:

$$\int_V (\bar{b}_i + \Delta\bar{b}_i)\delta u_i\,dV + \int_{S_T}(\bar{T}_i + \Delta\bar{T}_i)\delta u_i\,dS = \int_V -\xi_{ij}\frac{1}{2}\delta(u_{i,j}+u_{j,i})\,dV. \tag{27}$$

$$\xi_{ij} = \beta_{hk}\Delta\tilde{\lambda}_\alpha\frac{\partial^2\Phi_\alpha}{\partial\sigma_{hk}\,\partial\sigma_{rs}}(\tilde{\tau}_{lm})E_{rsij} - \Delta\tilde{\varepsilon}_{hk}E_{ijhk} - \bar{\sigma}_{ij}, \tag{28}$$

$$\alpha_{ij}(\tilde{\varepsilon}^{p}_{lm}) + \xi_{ij} + \beta_{ij} + \beta_{hk}\Delta\tilde{\lambda}_\alpha\frac{\partial^2\Phi_\alpha}{\partial\sigma_{hk}\,\partial\sigma_{rs}}(\tilde{\tau}_{lm})\frac{\partial\alpha_{rs}}{\partial\varepsilon^{p}_{ij}}(\tilde{\varepsilon}^{p}_{lm}) = 0, \tag{29}$$

$$\eta_\alpha = \beta_{ij}\frac{\partial\Phi_\alpha}{\partial\sigma_{ij}}(\tilde{\tau}_{lm}) - Y_\alpha(\tilde{\lambda}_\beta), \tag{30}$$

$$2(\eta\tilde{\ell})_\alpha = 0. \tag{31}$$

Equation (27), by the virtual work principle, expresses the equilibrium of the tensor field $-\xi_{ij}$ (if interpreted as stresses) under the loads at the end of the step.

Now substitute eqn (28) into (29) and rearrange; denoting by I_{ijhk} the identity tensor:

$$\left[I_{ijhk}+\Delta\tilde{\lambda}_{\alpha}\frac{\partial^2\Phi_{\alpha}}{\partial\sigma_{hk}\,\partial\sigma_{rs}}(\bar{\tau}_{lm})\cdot\left(E_{rsij}+\frac{\partial\alpha_{rs}}{\partial\varepsilon^{p}_{ij}}(\bar{\varepsilon}^{p}_{lm})\right)\right]\beta_{hk}=\bar{\sigma}_{ij}+E_{ijhk}\Delta\tilde{\varepsilon}^{e}_{hk}-\alpha_{ij}(\bar{\varepsilon}^{p}_{lm}). \quad (32)$$

Interpret eqn (32) as a linear system in the unknowns β_{hk}. Noting, by eqn (25), that the r.h.s. is $\bar{\tau}_{ij}$ and using the identity (14b) due to the hypothesis (b) Section 2.4, one realizes that this system admits the solution:

$$\beta_{ij}=\bar{\tau}_{ij}=\bar{\sigma}_{ij}+E_{ijhk}\Delta\tilde{\varepsilon}^{e}_{hk}-\alpha_{ij}(\bar{\varepsilon}^{p}_{lm}). \quad (33)$$

This solution is unique. In fact, the identity tensor and the elastic tensor are positive definite; the Jacobian of α_{rs} (ε^{p}_{hk}) is positive semi-definite because of the convexity hypothesis (e) in Section 2.4; the Hessian of Φ_{α} too is so, because of the convexity hypothesis (a). Therefore, pre-multiplying both sides by the inverse of the positive definite tensor in round brackets, eqn (32) is transformed into an equivalent system, whose coefficient matrix is clearly nonsingular.

Substituting (33) into eqn (28), by virtue of identity (14b) again, we obtain:

$$\xi_{ij}=-\bar{\sigma}_{ij}-E_{ijhk}\Delta\tilde{\varepsilon}^{e}_{hk}=-\tilde{\sigma}_{ij}. \quad (34)$$

Substitute (34) into eqn (27) which holds for any δu_i: this turns out to express, by the virtual work principle, the equilibrium of the stress field $\tilde{\sigma}_{ij}$ with the load at step end, namely the fulfilment of eqn (11).

Taking into account (33) in eqn (30), η_{α} is recognized as the yield function ϕ_{α}, eqn (7a), in view of the homogeneity hypothesis (b) and of the consequent eqn (14a).

As a consequence, eqn (31) becomes equivalent to the complementarity equation (9), since the constraint (23b), expressed as $\Delta\lambda_{\alpha}=\ell^{2}_{\alpha}$, holds at the minimum ($\sim$) as well.

At the minimum, the second variation $\delta^{(2)}L$ of functional L, is non-negative which implies, in particular:

$$-\int_V[\eta(\delta\ell)^2]_{\alpha}\,\mathrm{d}V\geqslant 0. \quad (35)$$

Therefore, $\eta_{\alpha}\leqslant 0$ and hence, $\phi_{\alpha}\leqslant 0$.

At this stage, it can be concluded that the constraints (22)–(23), which include geometric compatibility, together with the constrained minimization of Ω imply the fulfilment of all the relations (6)–(11) governing the finite-step problem, subject to the restrictions of Section 2.4.

Proof of necessity. Capped symbols will denote quantities pertaining to a solution of the finite-step elastoplastic problem (6)–(11); primed symbols will mark quantities pertaining to fields which satisfy the constraints (22)–(23) ("feasible solution" for the optimization problem). Denoting by unbarred symbols values at the end of the step (e.g. $\lambda'=\bar{\lambda}_{\alpha}+\Delta\lambda'_{\alpha}$), after some trivial algebra we can write:

$$\Omega'-\hat{\Omega}=\int_V\frac{1}{2}(\Delta\varepsilon^{e'}_{ij}-\Delta\hat{\varepsilon}^{e}_{ij})E_{ijhk}\cdot(\Delta\varepsilon^{e'}_{hk}-\Delta\hat{\varepsilon}^{e}_{hk})\,\mathrm{d}V+\int_V[\Pi(\lambda'_{\alpha})-\Pi(\hat{\lambda}_{\alpha})]\,\mathrm{d}V$$

$$+\int_V[\Gamma(\varepsilon^{p'}_{ij})-\Gamma(\hat{\varepsilon}^{p}_{ij})]\,\mathrm{d}V+\int_V\Delta\hat{\varepsilon}^{e}_{ij}E_{ijhk}\cdot(\Delta\varepsilon^{e'}_{hk}-\Delta\hat{\varepsilon}^{e}_{hk})\,\mathrm{d}V+\int_V\bar{\sigma}_{ij}\cdot(\Delta\varepsilon^{e'}_{hk}-\Delta\hat{\varepsilon}^{e}_{hk})\,\mathrm{d}V$$

$$-\int_V b_i\cdot(\Delta u'_i-\Delta\hat{u}_i)\,\mathrm{d}V-\int_{S_T}T_i\cdot(\Delta u'_i-\Delta\hat{u}_i)\,\mathrm{d}S. \quad (36)$$

Note now that: the first integrand is non-negative; because of the convexity of Π and Γ, respectively, the second and third integrands are bounded below by the linear term of their Taylor expansions around the solution, in view of an inequality similar to (12); the algebraic sum of the last four integrals can be replaced by a single integral making use of a virtual work equation (since the actual static quantities at the step end are equilibrated and the differences of kinematic quantities are compatible if plastic strain increments are added). Therefore, we may write:

$$\Omega'-\hat{\Omega} \geqslant \int_V Y_\alpha(\hat{\lambda}_\beta)\cdot(\Delta\lambda'_\alpha-\Delta\hat{\lambda}_\alpha)\,dV+\int_V \alpha_{ij}(\hat{\varepsilon}^p_{rs})\cdot(\Delta\varepsilon'^p_{ij}-\Delta\hat{\varepsilon}^p_{ij})\,dV-\int_V \hat{\sigma}_{ij}\cdot(\Delta\varepsilon'^p_{ij}-\Delta\hat{\varepsilon}^p_{ij})\,dV. \tag{37}$$

Substituting the usual expressions for plastic strain increments, using position (25) and rearranging, the inequality (37) becomes:

$$\Omega'-\hat{\Omega} \geqslant -\int_V \frac{\partial\Phi_\alpha}{\partial\sigma_{ij}}(\tau'_{rs})\hat{\tau}_{ij}\Delta\lambda'_\alpha\,dV+\int_V Y_\alpha(\hat{\lambda}_\beta)\Delta\lambda'_\alpha\,dV+\int_V\left[-Y_\alpha(\hat{\lambda}_\beta)+\frac{\partial\Phi_\alpha}{\partial\sigma_{ij}}(\hat{\tau}_{rs})\hat{\tau}_{ij}\right]\Delta\hat{\lambda}_\alpha\,dV. \tag{38}$$

The last integrand vanishes because of the complementarity eqn (9), since the factor in brackets is recognized as $\hat{\phi}_\alpha$. The first integral can be replaced by its lower bound, provided by eqn (12) and evaluated with $\tau_{ij}=\hat{\tau}_{ij}$ if ϕ_α is convex (hyp. a), taking into account eqn (14a) if it is also homogeneous of first-order (hyp. b). Then we have:

$$\Omega'-\hat{\Omega} \geqslant \int_V [Y_\alpha(\hat{\lambda}_\beta)-\Phi_\alpha(\hat{\sigma}_{ij}-\alpha_{ij}(\hat{\varepsilon}^p_{hk}))]\Delta\lambda'_\alpha\,dV. \tag{39}$$

The factor in brackets equals $-\hat{\phi}_\alpha \geqslant 0$, while $\Delta\lambda'_\alpha \geqslant 0$. Therefore, $\Omega' \geqslant \hat{\Omega}$, namely the value attained by the functional Ω at the finite-step solution, turns out to be the (absolute) minimum over the feasible domain defined by constraints (22)–(23).

3.2. *A theorem in the increments of plastic strains and internal variables*

The linear elastic stress response $\Delta\bar{\sigma}^e_{ij}$ to the given external action changes over Δt is now assumed to be preliminarily computed. For the remaining computations in the step, $\Delta\bar{\sigma}^e_{ij}$ represents data containing all information on the load increments. We denote by $\Delta\sigma^p_{ij}$ and $\Delta\varepsilon^{ep}_{ij}$ the (self-equilibrated) stresses and corresponding elastic strains generated in the body by the (unknown) plastic strains $\Delta\varepsilon^p_{ij}$, should they act as imposed strains on the body assumed as linear elastic in the absence of any external action (hence, with homogeneous boundary conditions). In principle, one can write:

$$\Delta\sigma^p_{ij}=\Delta\sigma^p_{ij}(\Delta\varepsilon^p_{rs})=\int_V Z_{ijhk}(x_r,\xi_r)\Delta\varepsilon^p_{hk}(\xi_r)\,dV. \tag{40}$$

The tensor-valued Green's function in (40) is analytically known or obtainable in few cases, e.g. for the isotropic homogeneous elastic domain (for which it can be derived from Kelvin's fundamental solution of the Navier equation, see Bui, 1978). However, conventional discretizations in space reduce the linear integral operator in (40) to a matrix. Its discrete version by, say, compatible finite element models, preserves the symmetry and negative semi-definiteness which are the essential features of the above operator (and can be easily proved by the virtual work principle, see e.g. Maier, 1970b). These facts confer computational interest to the following statement, which will be proved as a consequence of the kinematic theorem of Section 3.1.

Proposition 2. Under the constitutive restrictions (a)–(e) of Section 2.4, the finite-step b.v. problem (6)–(11) is equivalent to the optimization problem:

$$\min\left\{\bar{\Omega}(\Delta\varepsilon^p_{ij},\Delta\lambda_\alpha) \equiv -\frac{1}{2}\int_V\int_V \Delta\varepsilon^p_{ij}(x_r)Z_{ijhk}(x_r,\xi_s)\Delta\varepsilon^p_{hk}(\xi_s)\,\mathrm{d}V_x\,\mathrm{d}V_\xi\right.$$

$$\left.+\int_V[\Pi(\bar{\lambda}_\alpha+\Delta\lambda_\alpha)+\Gamma(\bar{\varepsilon}^p_{ij}+\Delta\varepsilon^p_{ij})]\,\mathrm{d}V-\int_V(\bar{\sigma}^e_{ij}+\Delta\bar{\sigma}^e_{ij})\Delta\varepsilon^p_{ij}\,\mathrm{d}V-\int_V\bar{\sigma}^p_{ij}\Delta\varepsilon^p_{ij}\,\mathrm{d}V\right\} \quad (41)$$

subject to:

$$\Delta\varepsilon^p_{ij}=\frac{\partial\Phi_\alpha}{\partial\tau_{ij}}(\tau_{hk})\Delta\lambda_\alpha,\quad \Delta\lambda_\alpha\geqslant 0 \quad (42a, b)$$

where

$$\tau_{ij}\equiv\bar{\sigma}_{ij}+\Delta\bar{\sigma}^e_{ij}+\int_V Z_{ijhk}(x_r,\xi_s)\Delta\varepsilon^p_{hk}(\xi_s)\,\mathrm{d}V_\xi-\alpha_{ij}(\bar{\varepsilon}^p_{rs}+\Delta\varepsilon^p_{rs}). \quad (43)$$

Proof. Substitute into the functional of Proposition 1, the new interpretation of increments:

$$\Delta\varepsilon^e_{ij}=\Delta\varepsilon^{ee}_{ij}+\Delta\varepsilon^{ep}_{ij}\equiv C_{ijhk}(\Delta\bar{\sigma}^e_{hk}+\Delta\sigma^p_{hk}). \quad (44)$$

where C_{ijhk} is the elastic compliance tensor.

Write virtual work equations which exploit equilibrium and compatibility of the two fictitious elastic responses (to external action and plastic strain increments), such as:

$$\int_V b_i\Delta u_i\,\mathrm{d}V+\int_{S_T}T_i\Delta u_i\,\mathrm{d}S+\int_{S_u}(\bar{\sigma}^e_{ij}+\Delta\bar{\sigma}^e_{ij})n_j\Delta\bar{u}_i\,\mathrm{d}S$$

$$=\int_V(\bar{\sigma}^e_{ij}+\Delta\bar{\sigma}^e_{ij})(\Delta\varepsilon^{ee}_{ij}+\Delta\varepsilon^{ep}_{ij}+\Delta\varepsilon^p_{ij}+\Delta\bar{\theta}_{ij})\,\mathrm{d}V; \quad (45)$$

$$\int_V\sigma^p_{ij}\cdot(\Delta\varepsilon^{ep}_{ij}+\Delta\varepsilon^p_{ij})\,\mathrm{d}V=0. \quad (46)$$

Using these equations and eqn (40) and dropping constant terms, functional (21) reduces to (41). The compatibility constraints (22) no longer intervene as they are fulfilled by the very definitions of the two addends which appear in (44). This implies that displacements are no longer among the variables.

As for the constraints (23a) and (23b), they remain unaltered except that eqn (40) is taken into account in the argument τ_{ij}.

4. DUAL EXTREMUM PROPERTIES OF FINITE-STEP HOLONOMIC SOLUTIONS

4.1. *Static (complementary energy) theorem*

4.1.1. Referring again to the step-holonomic b.v. problem (6)–(11), we prove below the following statement.

Proposition 3. Consider the optimization problem:

$$\min\left\{\Omega_c(\Delta\sigma_{ij},\Delta\lambda_\alpha,\Delta\varepsilon^p_{ij})\equiv\frac{1}{2}\int_V\Delta\sigma_{ij}C_{ijhk}\Delta\sigma_{hk}\,\mathrm{d}V+\int_V\Pi_c(\bar{\lambda}_\alpha+\Delta\lambda_\alpha)\,\mathrm{d}V+\int_V\Gamma_c(\bar{\varepsilon}^p_{ij}+\Delta\varepsilon^p_{ij})\,\mathrm{d}V\right.$$

$$\left.-\int_V Y_\alpha(\bar{\lambda}_\beta+\Delta\lambda_\beta)\bar{\lambda}_\alpha\,\mathrm{d}V-\int_V\alpha_{hk}(\bar{\varepsilon}^p_{ij}+\Delta\varepsilon^p_{ij})\bar{\varepsilon}^p_{hk}\,\mathrm{d}V+\int_V\Delta\sigma_{ij}\Delta\bar{\theta}_{ij}\,\mathrm{d}V-\int_{S_u}\Delta\sigma_{ij}n_j\Delta\bar{u}_i\,\mathrm{d}S\right\}, \quad (47)$$

where Π_c and Γ_c denote the following functions:

$$\Pi_c \equiv Y_\alpha(\lambda_\beta)\lambda_\alpha - \Pi; \quad \Gamma_c \equiv \alpha_{ij}(\varepsilon^p_{rs})\varepsilon^p_{ij} - \Gamma \tag{48}$$

subject to the relations:

$$\Delta\sigma_{ij,i} + \Delta\bar{b}_j = 0 \quad \text{in } V, \quad n_i\Delta\sigma_{ij} = \Delta\bar{T}_j \quad \text{on } S_T \tag{49a, b}$$

$$\phi_\alpha = \Phi_\alpha(\bar{\sigma}_{ij} + \Delta\sigma_{ij} - \alpha_{ij}(\bar{\varepsilon}^p_{rs} + \Delta\varepsilon^p_{rs})) - Y_\alpha(\bar{\lambda}_\beta + \Delta\lambda_\beta) \leqslant 0 \quad \text{in } V. \tag{50}$$

When the constitutive restrictions (a)–(e) of Section 2.4 hold, any solution to the b.v. problem (6)–(11) minimizes the functional Ω_c, subject to the constraints (49)–(50). Also the converse is true (sufficient condition) when hypotheses (a)–(e) are fulfilled with strict convexity of Π and Γ.

Proof of sufficiency. Being understood that equilibrium (49) is implicitly complied with by any stress field considered (so that $\delta\sigma_{hk}$ will be self-equilibrated), we write the augmented functional:

$$L_c = \Omega_c + \int_V \eta_\alpha \cdot (\phi_\alpha + \ell^2_\alpha)\,\mathrm{d}V. \tag{51}$$

Compute and set to zero its first variation due to variations $\delta\sigma_{ij}$, $\delta\lambda_\alpha$, $\delta\varepsilon^p_{ij}$, $\delta\ell_\alpha$ and $\delta\eta_\alpha$ around the minimum marked by $\sim$; thereafter, take into account eqn (50) written as $\phi_\alpha + \ell^2_\alpha = 0$, and, finally rearrange to obtain:

$$\begin{aligned}\delta^{(1)}L_c = &\int_V \left[C_{ijhk}\Delta\tilde{\sigma}_{ij} + \Delta\bar{\theta}_{hk} + \eta_\alpha \frac{\partial\Phi_\alpha}{\partial\sigma_{hk}}(\tilde{\tau}_{ij})\right]\delta\sigma_{hk}\,\mathrm{d}V \\ &- \int_{S_u} \Delta\bar{u}_h n_k \delta\sigma_{hk}\,\mathrm{d}S + \int_V \left[\frac{\partial Y_\alpha}{\partial\lambda_\beta}(\tilde{\lambda}_\gamma)(\tilde{\lambda}_\beta - \bar{\lambda}_\beta - \eta_\beta)\right]\delta\lambda_\alpha\,\mathrm{d}V \\ &+ \int_V \left[\frac{\partial\alpha_{ij}}{\partial\varepsilon^p_{hk}}(\tilde{\varepsilon}^p_{rs})\left(\tilde{\varepsilon}^p_{hk} - \bar{\varepsilon}^p_{hk} - \frac{\partial\Phi_\alpha}{\partial\sigma_{hk}}(\tilde{\tau}_{rs})\eta_\alpha\right)\right]\delta\varepsilon^p_{ij}\,\mathrm{d}V + 2\int_V [\eta\ell]_\alpha \delta\ell_\alpha\,\mathrm{d}V = 0.\end{aligned} \tag{52}$$

The stationarity of L_c at the minimum entails the following consequences:

(a) The kinematic quantities involved in the first two integrals are geometrically compatible, due to the virtual work principle because $\delta\sigma_{hk}$ is self-equilibrated, but otherwise arbitrary.

(b) The expressions in brackets in the last three integrals vanish everywhere in V. This implies, respectively [subject to: strict convexity of Π, i.e. nonsingularity of its Hessian, as for (53); strict convexity of Γ as for (54)]:

$$\eta_\beta = \Delta\tilde{\lambda}_\beta \tag{53}$$

$$\Delta\tilde{\varepsilon}^p_{ij} = \frac{\partial\Phi_\alpha}{\partial\sigma_{ij}}\Delta\tilde{\lambda}_\alpha \tag{54}$$

$$[\tilde{\phi}\Delta\tilde{\lambda}]_\alpha = 0. \tag{55}$$

The non-negativeness of the second variation $\delta^{(2)}L_c$, of the Lagrangian implies, in particular, that

$$\eta_\beta \geqslant 0. \tag{56}$$

Thus, it can be concluded that the constrained minimization of Ω_c leads to the fulfilment of all those relations governing the b.v. problem, which are not expressed by the constraints (49) and (50).

Proof of necessity. Capped symbols will mark quantities which solve the step problem (6)–(11), starred symbols quantities which are feasible with respect to constraints (49)–(50); barred and unbarred symbols for cumulative quantities refer to the starting and the end instant of the step, respectively (e.g. $\lambda_\alpha^* = \bar{\lambda}_\alpha + \Delta\lambda_\alpha^*$). Using eqns (47), after some easy manipulations, we may write:

$$\Omega_c^* - \hat{\Omega}_c = \int_V \frac{1}{2}(\Delta\sigma_{ij}^* - \Delta\hat{\sigma}_{ij})C_{ijhk}\cdot(\Delta\sigma_{hk}^* - \Delta\hat{\sigma}_{hk})\,\mathrm{d}V + \int_V [\Pi(\hat{\lambda}_\alpha) - \Pi(\lambda_\alpha^*)]\,\mathrm{d}V$$
$$+ \int_V [\Gamma(\hat{\varepsilon}_{ij}^p) - \Gamma(\varepsilon_{ij}^{p*})]\,\mathrm{d}V + \int_V [Y_\alpha(\lambda_\beta^*)(\lambda_\alpha^* - \bar{\lambda}_\alpha) - Y_\alpha(\hat{\lambda}_\beta)(\hat{\lambda}_\alpha - \bar{\lambda}_\alpha)]\,\mathrm{d}V$$
$$+ \int_V [\alpha_{ij}(\varepsilon_{rs}^{p*})(\varepsilon_{ij}^{p*} - \bar{\varepsilon}_{ij}^p) - \alpha_{ij}(\hat{\varepsilon}_{rs}^p)(\hat{\varepsilon}_{ij}^p - \bar{\varepsilon}_{ij}^p)]\,\mathrm{d}V$$
$$+ \int_V (\Delta\hat{\sigma}_{ij}C_{ijhk} + \Delta\bar{\theta}_{hk})(\Delta\sigma_{hk}^* - \Delta\hat{\sigma}_{hk})\,\mathrm{d}V - \int_{S_u} (\Delta\sigma_{hk}^* - \Delta\hat{\sigma}_{hk})n_h\Delta\bar{u}_k\,\mathrm{d}S. \quad (57)$$

The following circumstances should be noted in eqn (57): the first integrand is non-negative; because of the assumed convexity of $\Pi(\lambda_\alpha)$ and $\Gamma(\varepsilon_{ij}^p)$ (which does not imply convexity of Π_c and Γ_c), the second and third integrands are not less than the linear terms of their Taylor expansions around λ_α^* and ε_{ij}^{p*}, respectively. Since $\Delta\sigma_{ij}^* - \Delta\hat{\sigma}_{ij}$ represents a self-equilibrated field and $\Delta\hat{\varepsilon}_{hk}$ and $\Delta\hat{u}_i$ are compatible, a virtual work equation permits one to replace the two last integrals by a single integral involving $\Delta\hat{\varepsilon}_{ij}^p$. Therefore, eqn (57) gives rise to the inequality:

$$\Omega_c^* - \hat{\Omega}_c \geqslant \int_V Y_\beta(\lambda_\alpha^*)(\hat{\lambda}_\beta - \lambda_\beta^*)\,\mathrm{d}V + \int_V \alpha_{ij}(\varepsilon_{rs}^{p*})(\hat{\varepsilon}_{ij}^p - \varepsilon_{ij}^{p*})\,\mathrm{d}V + \int_V [Y_\alpha(\lambda_\beta^*)\Delta\lambda_\alpha^* - Y_\alpha(\hat{\lambda}_\beta)\Delta\hat{\lambda}_\alpha]\,\mathrm{d}V$$
$$+ \int_V [\alpha_{ij}(\varepsilon_{rs}^{p*})\Delta\varepsilon_{ij}^{p*} - \alpha_{ij}(\hat{\varepsilon}_{rs}^p)\Delta\hat{\varepsilon}_{ij}^p]\,\mathrm{d}V - \int_V (\Delta\sigma_{hk}^* - \Delta\hat{\sigma}_{hk})\Delta\hat{\varepsilon}_{hk}^p\,\mathrm{d}V. \quad (58)$$

By rearranging, expressing $\Delta\hat{\varepsilon}_{hk}^p$ in the form (8a) with normality ($\psi_\alpha = \phi_\alpha$), and taking into account eqns (12) and (14a) as in the necessity proof of Proposition 1, one obtains successively:

$$\Omega_c^* - \hat{\Omega}_c \geqslant -\int_V [Y_\beta(\hat{\lambda}_\alpha) - Y_\beta(\lambda_\alpha^*)]\Delta\hat{\lambda}_\beta\,\mathrm{d}V - \int_V [\alpha_{ij}(\hat{\varepsilon}_{rs}^p) - \alpha_{ij}(\varepsilon_{rs}^{p*})]\Delta\hat{\varepsilon}_{ij}^p\,\mathrm{d}V$$
$$+ \int_V (\hat{\sigma}_{ij} - \sigma_{ij}^*)\Delta\hat{\varepsilon}_{ij}^p\,\mathrm{d}V \geqslant \int_V \left\{[\hat{\sigma}_{ij} - \alpha_{ij}(\hat{\varepsilon}_{rs}^p)]\frac{\partial\Phi_\beta}{\partial\tau_{ij}}(\hat{\tau}_{rs}) - Y_\beta(\hat{\lambda}_\alpha)\right\}\Delta\hat{\lambda}_\beta\,\mathrm{d}V$$
$$-\int_V \left\{[\sigma_{ij}^* - \alpha_{ij}(\varepsilon_{rs}^{p*})]\frac{\partial\Phi_\beta}{\partial\tau_{ij}}(\tau_{rs}^*) - Y_\beta(\lambda_\alpha^*)\right\}\Delta\hat{\lambda}_\beta\,\mathrm{d}V. \quad (59a, b)$$

In (59b), the expressions in { } are recognized as the yield function $\hat{\phi}_\beta$ in the solution (former integral) and as ϕ_β^* for a feasible field (latter integral). Therefore, the former integral vanishes and the latter is nonpositive in view of eqn (50), since $\Delta\hat{\lambda}_\beta \geqslant 0$.

4.1.2. The restriction mentioned at the end of Proposition 3 for the validity of the sufficient condition, rules out the important case of ideal plasticity. This is characterized by the constitutive features: Y_α = constant and $\alpha_{ij} = 0$, which imply $\Pi_c = 0$ and $\Gamma_c = 0$. By virtue of such specializations, Proposition 3 can be rephrased as follows.

Proposition 3*. The optimization problem

$$\min\left\{\Omega_c(\Delta\sigma_{ij}) = \frac{1}{2}\int_V \Delta\sigma_{ij}C_{ijhk}\Delta\sigma_{hk}\,\mathrm{d}V + \int_V \Delta\sigma_{ij}\Delta\bar{\theta}_{ij}\,\mathrm{d}V - \int_{S_u}\Delta\sigma_{ij}n_i\Delta\bar{u}_j\,\mathrm{d}S\right\}, \quad (60)$$

subject to:

$$\Delta\sigma_{ij,i}+\Delta\bar{b}_j = 0 \quad \text{in } V; \quad n_i\Delta\sigma_{ij} = \Delta\bar{T}_j \quad \text{on } S_T \quad (61\text{a, b})$$

$$\phi_\alpha = \Phi_\alpha(\bar{\sigma}_{ij}+\Delta\sigma_{ij}) - Y_\alpha \leqslant 0 \quad \text{in } V, \quad (62)$$

is equivalent to the b.v. problem (6)–(11) specialized to ideal plasticity.

Proof of sufficiency. Let us follow once again the path of reasoning which led to the sufficiency part of Proposition 3: note that $\Delta\lambda_\alpha$ and $\Delta\varepsilon^p_{ij}$ do not show up in problems (60)–(62) and hence, in the first variation (52) of the specialized Lagrangian functional, the third and fourth addends (volume integrals) are missing. Therefore, the requirements that Hessians of $\Pi(\lambda_\alpha)$ and $\Gamma(\varepsilon^p_{ij})$ contained in them be nonsingular (i.e. Π and Γ strictly convex) are unnecessary. However, eqns (53) and (54) are still legitimate and the whole set of the relations governing the b.v. problem is still recovered. In fact, the compatibility arises again from the first and second addends in eqn (52) by the virtual work principle; the complementarity (55) from the last addend in (52); the sign constraints (56) from the non-negativeness of the second variations as before. Interpreting the Lagrangian multipliers η_α as plastic multiplier increments $\Delta\lambda_\alpha$, eqn (53), we can write eqn (54).

Proof of necessity holds unaltered with self-evident specializations.

Remarks. It is worth stressing that the main difference between the proofs of sufficiency in Proposition 3 and Proposition 3* rests on the fact that in the latter, ideal plasticity case, stresses only (not also $\Delta\varepsilon^p_{ij}$) appear in the argument τ_{ij} in eqn (52).

Proposition 3* can be recognized as an extension to the finite increment problem of Haar–Kàrmàn theorem (Martin, 1975), to which it reduces if the step starts from the unstressed state ($\bar{\sigma}_{ij} = 0$).

4.2. *A dual extremum property of plastic strains and internal variables*

Interpreting again the elastic–plastic response $\Delta\sigma_{ij}$ as the sum of the linear–elastic stress response $\Delta\bar{\sigma}^e_{ij}$ and the self-equilibrated stresses $\Delta\sigma^p_{ij}$ generated by plastic strains $\Delta\varepsilon^p_{ij}$, we prove below the following extremum theorem dual to Proposition 2.

Proposition 4. Consider the problem:

$$\min\left\{\Omega_c(\Delta\varepsilon^p_{ij},\Delta\lambda_\alpha) \equiv -\frac{1}{2}\int_V\int_V \Delta\varepsilon^p_{ij}(x_r)Z_{ijhk}(x_r,\xi_s)\Delta\varepsilon^p_{hk}(\xi_s)\,\mathrm{d}V_x\,\mathrm{d}V_\xi + \int_V [\Pi_c(\bar{\lambda}_\alpha+\Delta\lambda_\alpha)\right.$$

$$\left. +\Gamma_c(\bar{\varepsilon}^p_{ij}+\Delta\varepsilon^p_{ij})]\,\mathrm{d}V - \int_V Y_\beta(\bar{\lambda}_\alpha+\Delta\lambda_\alpha)\bar{\lambda}_\beta\,\mathrm{d}V - \int_V \alpha_{hk}(\bar{\varepsilon}^p_{ij}+\Delta\varepsilon^p_{ij})\bar{\varepsilon}^p_{hk}\,\mathrm{d}V\right\}, \quad (63)$$

subject to the relation:

$$\phi_\alpha = \Phi_\alpha(\tau_{ij}) - Y_\alpha(\bar{\lambda}_\beta+\Delta\lambda_\beta) \leqslant 0, \quad (64)$$

where τ_{ij} is defined by eqn (43) and contains the data of the external action step through the pre-calculated elastic stresses $\Delta\bar{\sigma}^e_{ij}$.

Under the constitutive restrictions (a)–(e) any solution to the boundary value problem (6)–(11) solves the minimization problem (63)–(64), and the converse is true if strict convexity of Π and Γ is assumed.

Proof. Splitting again the actual response into two addends, as in eqn (43), use eqn

(45) again and the further virtual work equation

$$\int_V \Delta\sigma^p_{ij}(C_{ijrs}\Delta\bar{\sigma}^e_{rs}+\Delta\bar{\theta}_{ij})\,\mathrm{d}V = \int_{S_u} \Delta\sigma^p_{ij}n_i\Delta\bar{u}_j\,\mathrm{d}S. \tag{65}$$

Thus, taking into account eqn (40) and dropping constant terms, the functional (47) of Proposition 3 is easily transformed into (63). The equilibrium constraints (49) are *a priori* fulfilled by both fictitious linear responses (to external action and plastic strain increments) in view of their very definition; therefore these constraints can be dropped. Condition (50) becomes constraint (64) with the integral expression (40) of self-stresses induced by plastic strains.

Remark. The complementary plastic work can be given the following expression, if the constitutive restrictions (b)–(d) hold and eqns (48) are used:

$$W^p_c(\bar{t}+\Delta t) \equiv \int_0^{\bar{t}} \varepsilon^p_{ij}(t')\dot{\sigma}_{ij}(t')\,\mathrm{d}t' + \int_{\bar{t}}^{\bar{t}+\Delta t} \varepsilon^p_{ij}(t')\dot{\sigma}_{ij}(t')\,\mathrm{d}t' = \Pi_c(\bar{\lambda}_\alpha+\Delta\lambda_\alpha)$$

$$+\Gamma_c(\bar{\varepsilon}^p_{ij}+\Delta\varepsilon^p_{ij})+\sigma_{ij}\bar{\varepsilon}^p_{ij} - Y_\alpha(\bar{\lambda}_\beta+\Delta\lambda_\beta)\bar{\lambda}_\alpha - \alpha_{ij}(\bar{\varepsilon}^p_{rs}+\Delta\varepsilon^p_{rs})\bar{\varepsilon}^p_{ij}. \tag{66}$$

This shows that the functions Π_c and Γ_c defined by eqns (48) represent the two addends into which the complementary plastic work can be split (as Π and Γ for W^p), provided the starting situation for the step coincides with the original undeformed state ($\bar{\varepsilon}^p_{ij} = 0$, $\bar{\lambda}_\alpha = 0$).

5. GENERAL EXTREMUM STATEMENTS FOR FINITE-STEP PROBLEMS

We will keep in what follows, only two of the constitutive restrictions of Section 2.4, namely: (b) the effective stresses are positively homogeneous functions of the order of one of their argument; (c) the flow rule is associative, i.e. the plastic potentials coincide with the yield functions. Note that no condition is imposed on the hardening rule and, in particular, softening is now admitted. The two statements given below rest on this weaker hypothesis basis.

Proposition 5. Under restrictions (b) and (c), the finite-step b.v. problem (6)–(11) is equivalent to the following constrained optimization, provided the minimum is zero (otherwise the b.v. problem has no solution):

$$\min\Big\{\bar{\Omega}(\Delta u_i,\Delta\lambda_\alpha,\Delta\varepsilon^p_{ij},\Delta\sigma_{ij}) \equiv \frac{1}{2}\int_V \Delta\varepsilon^e_{ij}E_{ijhk}\Delta\varepsilon^e_{hk}\,\mathrm{d}V + \int_V \bar{\sigma}_{ij}\Delta\varepsilon^e_{ij}\,\mathrm{d}V$$

$$-\int_V(\bar{b}_i+\Delta\bar{b}_i)\Delta u_i\,\mathrm{d}V - \int_{S_T}(\bar{T}_i+\Delta\bar{T}_i)\Delta u_i\,\mathrm{d}S + \frac{1}{2}\int_V \Delta\sigma_{ij}C_{ijhk}\Delta\sigma_{hk}\,\mathrm{d}V$$

$$+\int_V \Delta\varepsilon^p_{ij}\alpha_{ij}(\bar{\varepsilon}^p_{hk}+\Delta\varepsilon^p_{hk})\,\mathrm{d}V + \int_V \Delta\lambda_\alpha Y_\alpha(\bar{\lambda}_\beta+\Delta\lambda_\beta)\,\mathrm{d}V - \int_V \Delta\bar{\theta}\Delta\sigma_{ij}\,\mathrm{d}V$$

$$-\int_{S_u}\Delta\sigma_{ij}n_j\Delta\bar{u}_i\,\mathrm{d}S - \int_V \Delta\bar{\theta}_{ij}\bar{\sigma}_{ij}\,\mathrm{d}V - \int_{S_u}\bar{\sigma}_{ij}n_j\Delta\bar{u}_i\,\mathrm{d}S\Big\} = 0, \tag{67}$$

subject to:

$$\Delta\varepsilon^p_{ij} = \frac{\partial\Phi_\alpha}{\partial\sigma_{ij}}\Delta\lambda_\alpha;\quad \Delta\lambda_\alpha \geqslant 0;\quad \Phi_\alpha(\bar{\sigma}_{ij}+\Delta\sigma_{ij}-\alpha_{ij}(\bar{\varepsilon}^p_{hk}+\Delta\varepsilon^p_{hk})) - Y_\alpha(\bar{\lambda}_\beta+\Delta\lambda_\beta) \leqslant 0 \tag{68a,b,c}$$

$$\Delta\sigma_{ij} = E_{ijhk}\cdot(\tfrac{1}{2}(\Delta u_{h,k}+\Delta u_{k,h}) - \Delta\varepsilon^p_{hk} - \Delta\bar{\theta}_{hk})\quad \text{in } V;\quad \Delta u_i = \Delta\bar{u}_i\quad \text{on } S_u \tag{69a,b}$$

$$\Delta\sigma_{ij,i}+\Delta\bar{b}_j = 0 \quad \text{in } V; \quad \Delta\sigma_{ij}n_i = \Delta\bar{T}_j \quad \text{on } S_T. \tag{70a, b}$$

Proof. Using hypotheses (b) and subsequently (c), we can write, taking into account eqns (7a) and (25):

$$\Delta\lambda_\alpha\phi_\alpha = \Delta\lambda_\alpha\frac{\partial\Phi_\alpha}{\partial\tau_{ij}}(\tau_{rs})\tau_{ij}-\Delta\lambda_\alpha Y_\alpha(\bar{\lambda}_\beta+\Delta\lambda_\beta)$$

$$= \Delta\varepsilon^p_{ij}\cdot[\bar{\sigma}_{ij}+\Delta\sigma_{ij}-\alpha_{ij}(\bar{\varepsilon}^p_{rs}+\Delta\varepsilon^p_{rs})]-\Delta\lambda_\alpha Y_\alpha(\bar{\lambda}_\beta+\Delta\lambda_\beta). \tag{71a, b}$$

Rearranging and using (6), one obtains:

$$\Delta\lambda_\alpha\phi_\alpha = -\Delta\varepsilon^e_{ij}E_{ijrs}\Delta\varepsilon^e_{rs}-\Delta\varepsilon^p_{ij}\alpha_{ij}(\varepsilon^p_{hk})-(\bar{\sigma}_{ij}+\Delta\sigma_{ij})\Delta\theta_{ij}$$

$$-\Delta\varepsilon^e_{ij}\bar{\sigma}_{ij}-\Delta\lambda_\alpha Y_\alpha(\lambda_\beta)+(\bar{\sigma}_{ij}+\Delta\sigma_{ij})\Delta\varepsilon_{ij}. \tag{72}$$

Integration of the r.h.s. of (72) over the volume V and a virtual work equation to transform the last addend, leads to functional (67) with a reversed sign. Therefore, its minimization under constraints (68)–(70) means minimizing the integral over V of the non-negative integrand $-\phi_\alpha\Delta\lambda_\alpha$. If the minimum is zero, then the complementary condition (9) is satisfied everywhere and, since all other governing relations are expressed by the constraints, the b.v. problem is solved. Clearly, if the minimum is not zero, the set of governing relations (6)–(11) turns out to be incompatible. Conversely, any solution of (6)–(11) is clearly feasible for optimization (67)–(70) and makes its objective zero [and, hence, minimum, in view of the above noted meaning of functional (67)].

Proposition 6. When the elastic stress increments $\Delta\bar{\sigma}^e_{ij}$ have been determined, under the constitutive restrictions (b) and (d), the b.v. problem (6)–(11) is equivalent to the following minimization problem, if the minimum is zero (otherwise the b.v. problem is not solvable):

$$\min\left\{\bar{\Omega}(\Delta\varepsilon^p_{ij},\Delta\lambda_\alpha) \equiv -\int_V\int_V \Delta\varepsilon^p_{ij}(x_r)Z_{ijhk}(x_r,\xi_s)\Delta\varepsilon^p_{hk}(\xi_s)\,\mathrm{d}V_x\,\mathrm{d}V_\xi-\int_V(\bar{\sigma}_{ij}+\Delta\bar{\sigma}^e_{ij})\Delta\varepsilon^p_{ij}\,\mathrm{d}V\right.$$

$$\left.+\int_V\Delta\varepsilon^p_{ij}\alpha_{ij}(\bar{\varepsilon}^p_{hk}+\Delta\varepsilon^p_{hk})\,\mathrm{d}V+\int_V\Delta\lambda_\alpha Y_\alpha(\bar{\lambda}_\beta+\Delta\lambda_\beta)\,\mathrm{d}V\right\}, \tag{73}$$

subject to [eqn (43) defines τ_{ij}]:

$$\Delta\varepsilon^p_{ij} = \frac{\partial\Phi_\alpha}{\partial\tau_{ij}}(\tau_{hk})\Delta\lambda_\alpha,\quad \Delta\lambda_\alpha \geqslant 0,\quad \Phi_\alpha(\tau_{ij}) \leqslant Y_\alpha. \tag{74a, b, c}$$

Proof. As in Propositions 2 and 4, the superposition of the elastic responses to external actions (pre-determined as $\Delta\bar{\sigma}^e_{ij}$) and to plastic strain increments [through the integral operator (40) and relevant Green's kernel Z] makes compatibility and equilibrium *a priori* satisfied. The governing relations of the b.v. problem are imposed as constraints (74), except the complementarity condition (9); in view of the sign constraints of both factors, this can be enforced through the constrained minimization of:

$$\int_V -\phi_\alpha(\Delta\varepsilon^p_{ij},\Delta\lambda_\beta)\Delta\lambda_\alpha\,\mathrm{d}V. \tag{75}$$

This integral is easily transformed into the functional (73), using the constitutive hypotheses (b) and (c) alone, as done in the proof of Proposition 5.

Remark 1. The optimization problem concerned by Proposition 5 is easily seen to be related to those in Propositions 1 and 2 by means of two noteworthy links: (i) the functional $\bar{\Omega}$ is the sum of the two functionals Ω and Ω_c, and of a term which is constant within the step [last two integrals in (67)]; (ii) the feasible region defined by the constraints (68)–(70) is the intersection of the two regions defined by (22)–(23) and by (49)–(50), respectively. Similar connections relate Proposition 6 to Propositions 2 and 4. In other terms, it might be said that, when all the constitutive restrictions of Section 2.4 hold, Proposition 5 "splits" into the pair of dual, much simpler extremum characterizations stated in Propositions 1 and 3. Similarly, Proposition 6 "splits" into Propositions 2 and 4.

Remark 2. None of the constitutive restrictions of Section 2.4 are needed for the extremum characterization trivially achievable for the b.v. problem by minimizing to an integral like (75) subject to all governing relations except complementarity. Restrictions (b) and (c), i.e. normality and effective stress homogeneity of the order of one, as seen above by proving Propositions 5 and 6, confer to the functional to minimize a favorable, mechanically interpretable form and give rise to the links pointed out in the preceding remark.

Remark 3. The functionals envisaged in Propositions 5 and 6 are bounded below over the relevant feasible sets and hence, if these are not empty, their minimizations always have solutions. The fact that the minimum fails to vanish and hence, the b.v. problem has no solution, means that the body is incapable of carrying the current step of external actions. Clearly, such situations of collapse (whose identification will usually be improved by reducing the step amplitude) are characterized by lack of (bounded) solution in the context of the other Propositions 1–4.

6. CLOSING REMARKS

As a conclusion we briefly point out below the computational potentialities of the present results, their connections with earlier results via specialization and, finally, their possible extensions.

(a) If a suitable space discretization (e.g. by finite element modelling) is adopted, each one of the propositions pointed out in what precedes, reduces the finite-step problem arising from a backward-difference (or stepwise-holonomic) strategy for the approximate time integrations of the (nonlinear differential) relations governing elastic–plastic processes, to nonlinear programming. The nonlinear programming problem equivalent to the b.v. problem is generally nonconvex and not easy to solve. However, extremum characterizations of the solution to the finite-step elastic–plastic problem can lead to rigorous sufficient conditions for the convergence of iterative algorithms, devised to solve numerically that problem. With reference to Proposition 1, this circumstance is demonstrated in a parallel paper (Comi and Maier, 1990).

(b) A first kind of specializations of the present findings is obtained by assuming yield functions linear in the stresses and linear hardening. This constitutive piecewise-linearization reduces Propositions 1–6 to earlier extremum theorems (Maier, 1969, 1970a) which involve quadratic functionals and linear constraints and, in discrete versions, quadratic programming. A second way of specializing the developments expounded herein is to consider infinitesimal increments only ($\Delta t = \delta t \to 0$). This reduces again to quadratic–linear optimizations (or programming) and leads back to earlier rate theorems, precisely to statements proposed by Capurso (Proposition 1), Ceradini (Proposition 2), Capurso–Maier (Proposition 3) and Maier (Propositions 4, 5, 6). Surveys of such and other antecedents can be found e.g. in Martin (1975), Panagiotopoulos (1985), Cohn *et al.* (1979) and Maier *et al.* (1982). For perfectly-plastic materials and a single-step (starting from an unstressed state), Proposition 3 reduces to classical Haar–Kàrmàn theorem (see Martin, 1975). Recent contributions due to Martin and Reddy (1988), Reddy and Griffin (1986) and to Feijòo and Zouain (1988) seem to be related to Propositions 1 and 2, respectively, although they differ in approach, constitutive basis and path of reasoning.

(c) A relatively straightforward generalization of the present study would be to a broader class of elastic–plastic material models (primarily a class where the rates of the internal variables are distinct from the plastic multipliers intervening in the flow rules). No difficulty is expected in allowing for geometric effects on equilibrium, provided these are expressed by terms linearized in the displacement increments and containing the stresses at the beginning of the step. This would permit one to capture the influence of possible combined constitutive and geometrical instabilizing effects on the overall finite-step response (as on the rate response studied by Maier, 1971). A promising and a more challenging current development of this study is its extension to damage material models and to other constitutive laws not included in traditional elastoplasticity.

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Dedicated to Professor Erwin Stein on his 60th anniversary.